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The symmetric irreducible representations of SO_7 in $(SU_2)^3$ basis

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Abstract. In this paper, the symmetric irreducible representations of SO_7 in $(SU_2)^3$ basis are constructed. Some reduced matrix elements and closed expressions of simple isoscalar factors for $SO_7 \supset (SU_2)^3$ are obtained. Finally, an example is given to show how to obtain physical states from these mathematical ones.

1. Introduction

In nuclear and atomic physics the group chain $SU_7 \supset SO_7 \supset SO_3$ is very significant. For example, in nuclear physics this group chain is used to classify the octopole vibrations of the nucleus [1], and in atomic physics this group chain is used for classification of the f-electron, as has already been discussed by Racah [2] and Judd [3]. However, $SO_7 \supset SO_3$ is not simply reducible. In this reduction there are several missing labels, for which it is extremely difficult to find a simple physical interpretation. For this reason, De Mayer *et al* have examined the mathematical basis of SO_7 by the standard group-theoretical method [4, 5] and the shift-operator technique [6, 7]. But the explicit bases are not constructed in these papers, for which it is important to express the physical basis in terms of mathematical ones.

Generally, there is the reduction $SO_{2n+1} \rightarrow (SU_2)^n$ for integer n, where we restrict consideration to symmetric irreps of SO_{2n+1}. The first non-trivial case, SO_5, has been discussed by Kemmer *et al* [8] and Sun [9]. However, this reduction does not provide sufficient labels to lift degeneracy for the $n > 3$ case [4], so we will only treat symmetric irreps of SO_7 in this paper.

The intermediate steps in $SO_7 \supset (SU_2)^3$ may be analysed by using the embedding of semisimple complex Lie algebras in semisimple complex Lie algebras [10]. We find that only $(SU_2)^3$ is the maximal subalgebra of SO_7 in this reduction, i.e. there is no intermediate step in the reduction. The nine positive roots of SO_7 (B_3) are described [4] in an orthonormal basis $\{e_1, e_2, e_3\}$ as e_1, e_2, e_3, $e_1 \pm e_3$, $e_2 \pm e_3$. The three simple roots α_1, α_2, α_3 are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3$. In the root system of the algebra B_3 (SO_7) the appointment of three mutually orthogonal simple roots α'_1, α'_2 and α'_3 for the subalgebra $(SU_2)^3$ is not unique. It is clear that having defined α'_1, α'_2 and α'_3 in terms of e_1, e_2 and e_3, any formal permutation of the latter basis vectors leads to another acceptable definition of the simple roots. The branching rule is also independent of any particular choice. Our subalgebra choice based on the f-boson realisation of SO_7 will be given in the next section.

The complete inclusion $SO_7 \supset (SU_2)^3 \supset SU_2$ can also be considered. However, $(SU_2)^3 \supset SU_2$ is trivial because the basis of $(SU_2)^3 \supset SU_2$ are just the three SU_2 bases coupled to the final subgroup SU_2; the coupling coefficients needed here are CG coefficients of SU_2, which are well known. In addition, the physical basis $SO_7 \supset SO_3$ can easily be constructed from those non-physical bases $SO_7 \supset (SU_2)^3$.

In this paper, our discussion will be based on boson realisation of this group chain. The boson realisation of group symmetric irreps can be applied to a large class of simple Lie groups (see e.g. [11]).

In § 2 we will give an expression for the generators and Casimir operator of SO_7. In § 3 we will construct the explicit basis of SO_7. In § 4 we will given an example which shows how to obtain a physical state. The reduced matrix elements and some simple isoscalar factors for $SO_7 \supset (SU_2)^3$ will be given in § 5.

2. Generators

The branching rule of $SO_7 \supset (SU_2)^3$ is

$$SO_7 \supset SU_2^a \times SU_2^b \times SU_2^c \qquad (w, 0, 0) = a \times b \times c \tag{1}$$

where w is the seniority quantum number, and

$$2b = 2c = w - a - 2k \tag{2a}$$

$$a = 0, 1, 2, \ldots, w \tag{2b}$$

$$k = 0, 1, 2, \ldots, [\tfrac{1}{2}(w-a)]. \tag{2c}$$

Here the symbol $[x]$ denotes the maximum integer less than or equal to x.

We will use creation (annihilation) operators $f^\dagger_\mu$ (f_μ), $\mu = 0, \pm 1, \pm 2, \pm 3$, to construct the generators of SU_7, SO_7 and $(SU_2)^3$. First, we will define a set of uncoupling generators for the group chain (1) as follows [12]:

$$\chi_{\mu\nu} = f^\dagger_\mu \tilde{f}_\nu - f^\dagger_\nu \tilde{f}_\mu \qquad \mu, \nu = 0, \pm 1, \pm 2, \pm 3 \tag{3}$$

where $\tilde{f}_\mu = (-1)^{1+\mu} f_{-\mu}$. Obviously, $\chi_{\mu\nu} = -\chi_{\nu\mu}$, $\chi_{\mu\mu} = 0$, $(\chi_{\mu\nu})^\dagger = (-1)^{\mu+\nu}\chi_{-\nu-\mu}$, and in addition they satisfy the following commutation relation:

$$[\chi_{\mu\nu}, \chi_{\rho\sigma}] = (-1)^{1+\nu}\chi_{\mu\sigma}\delta_{\rho-\nu} + (-1)^{1+\sigma}\chi_{\nu\rho}\delta_{\mu-\sigma} + (-1)^{1+\nu}\chi_{\rho\mu}\delta_{\sigma-\nu} + (-1)^{1+\rho}\chi_{\sigma\nu}\delta_{\mu-\rho}. \tag{4}$$

Using $\chi_{\mu\nu}$, we can construct the generators of SO_7 and $(SU_2)^3$ as follows:

$$SU_2^a\!: \; a_0 = \chi_{1-1}, \; a_\pm = \pm\chi_{10} \tag{5a}$$

$$SU_2^b\!: \; b_0 = \tfrac{1}{2}(\chi_{3-3} - \chi_{2-2}), \; b_\pm = \pm(1/\sqrt{2})\chi_{\pm 3 \pm 2} \tag{5b}$$

$$SU_2^c\!: \; c_0 = \tfrac{1}{2}(\chi_{2-2} + \chi_{3-3}), \; c_\pm = \mp(1/\sqrt{2})\chi_{\pm 3 \mp 2}. \tag{5c}$$

The remaining generators of SO_7 can be put in the form of a tensor operator as given in table 1.

Table 1. Tensor operator $T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}$.

γ	α/β	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	1	$(1/\sqrt{2})\chi_{31}$	$(1/\sqrt{2})\chi_{-21}$
	0	$(1/\sqrt{2})\chi_{30}$	$(1/\sqrt{2})\chi_{-20}$
	-1	$(1/\sqrt{2})\chi_{3-1}$	$(1/\sqrt{2})\chi_{-2-1}$
$-\frac{1}{2}$	1	$(1/\sqrt{2})\chi_{12}$	$(1/\sqrt{2})\chi_{1-3}$
	0	$(1/\sqrt{2})\chi_{02}$	$(1/\sqrt{2})\chi_{0-3}$
	-1	$(1/\sqrt{2})\chi_{-12}$	$(1/\sqrt{2})\chi_{-1-3}$

Using (4), we can construct the commutation relations for these operators,

$$[A_+, A_-] = -A_0 \qquad [A_0, A_\pm] = \pm A_\pm \tag{6a}$$

where A_i can be taken as a_i, b_i or c_i, for $i = +1, -1, 0$, and

$$\begin{aligned}
&[a_\pm, T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}] = \mp\sqrt{\tfrac{1}{2}(1\mp\alpha)(2\pm\alpha)}\,T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\pm1\beta\gamma}\\
&[b_\pm, T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}] = \mp\sqrt{\tfrac{1}{2}(\tfrac{1}{2}\mp\beta)(\tfrac{3}{2}\pm\beta)}\,T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\pm1\gamma}\\
&[c_\pm, T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}] = \mp\sqrt{\tfrac{1}{2}(\tfrac{1}{2}\mp\gamma)(\tfrac{3}{2}\pm\gamma)}\,T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma\pm1}
\end{aligned} \tag{6b}$$

$$\begin{aligned}
&[T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}, T^{1\frac{1}{2}\frac{1}{2}}_{\alpha'\beta'\gamma'}]\\
&\quad = (1/\sqrt{2})\delta_{\alpha-\alpha'}\delta_{\beta-\beta'}(-1)^{\frac{3}{2}-\alpha-\beta}\langle\tfrac{1}{2}\gamma\tfrac{1}{2}\gamma'|1\gamma+\gamma'\rangle c_{\gamma+\gamma'}\\
&\qquad + (1/\sqrt{2})\delta_{\beta-\beta'}\delta_{\gamma-\gamma'}(-1)^{1-\gamma-\beta}\langle 1\alpha 1\alpha'|1\alpha+\alpha'\rangle a_{\alpha+\alpha'}\\
&\qquad + (1/\sqrt{2})\delta_{\alpha-\alpha'}\delta_{\gamma-\gamma'}(-1)^{\frac{3}{2}-\alpha-\gamma}\langle\tfrac{1}{2}\beta\tfrac{1}{2}\beta'|1\beta+\beta'\rangle b_{\beta+\beta'}.
\end{aligned} \tag{6c}$$

Using these generators, we can write the Casimir operator of SO_7 as

$$C_7 = a^2/2 + b^2 + c^2 + 2\sqrt{3}(T^{1\frac{1}{2}\frac{1}{2}} \times T^{1\frac{1}{2}\frac{1}{2}})^{(0)}_0. \tag{7}$$

3. Basis vectors

First, we introduce the following generators of SU(1, 1):

$$\begin{aligned}
S_+ &= \tfrac{1}{2}\sum_m (-1)^m f^\dagger_m f^\dagger_{-m}\\
S_- &= \tfrac{1}{2}\sum_m (-1)^m f_m f_{-m}\\
S_0 &= \tfrac{1}{4}\sum_m (f^\dagger_m f_m + f_m f^\dagger_m)
\end{aligned} \tag{8a}$$

they satisfy the following commutation relations:

$$[S_+, S_-] = -2S_0 \qquad [S_0, S_\pm] = \pm S_\pm. \tag{8b}$$

We can use the generators of SU(1, 1) to construct the tower of symmetric irreps of SU_7 and hence $SO_{(7)} \subset SU_{(7)}$. Let the basis vector of $SU_7 \supset SO_7$ be $|nw\Omega\rangle$, where Ω corresponds to other quantum numbers. The expectation value of the product S_+S_- is given by

$$\langle nw\Omega|S_+S_-|nw\Omega\rangle = S_0(S_0-1) - S(S-1) \tag{8c}$$

where $S(S-1)$ is the eigenvalue of the Casimir operator of SU(1, 1). Note that

$$S_0 = n/2+\tfrac{7}{4} \qquad S = w/2+\tfrac{7}{4}. \tag{8d}$$

The generators of SO_7 and of SU_7 leave the number of bosons invariant because they commute with the boson number operator. The generators of SU(1, 1) change the boson number by 0 or ± 2. The state $|nw\Omega\rangle$ with $n = w$ satisfies

$$S_-|ww\Omega\rangle = 0 \tag{8e}$$

because the state $|ww\Omega\rangle$ has the maximum seniority. The generators of SU(1, 1) group commute with SO_7 generators. Thus, the SU(1, 1) shift-up operator creates an invariant set of states with an even number of bosons (one state for each even n) from the ground state with no boson, and another invariant set of states with an odd number of bosons from the ground state with a single boson.

Thus, the state with n bosons which loads the symmetric irrep of SU_7 can be constructed by acting on $\frac{1}{2}(n-w)$ boson pairing operators S_+. It is easy to prove that the normalised state vector $|nw\Omega\rangle$ can be written as

$$|nw\Omega\rangle = \mathcal{N}(S_+)^{\rho}|ww\Omega\rangle \tag{9a}$$

where

$$\mathcal{N} = \left(\frac{(2w+5)!!}{2^{\rho}\rho!(2w+2\rho+5)!!}\right)^{1/2} \qquad n = w+2\rho. \tag{9b}$$

Next, we construct the state $|ww\Omega\rangle$. In the group chain (1), Ω can be written more explicitly as a, a_0; b, b_0, c_0. Obviously, the eigenstate of the operators C_7, a^2, a_0, b^2, b_0, c_0 with eigenvalues $\frac{1}{2}w(w+5)$, $a(a+1)$, a, $\frac{1}{2}(w-a)[\frac{1}{2}(w-a)+1]$, $\frac{1}{2}(w-a)$, $\frac{1}{2}(w-a)$ can be written as

$$|1\rangle = |ww;\, aa_0;\, bb_0c_0\rangle = N_1 f_1^{+a} f_3^{+w-a}|0\rangle \tag{10a}$$

with

$$N_1 = \left(\frac{1}{a!(w-a)!}\right)^{1/2}. \tag{10b}$$

Third, it can be proved that the operators $S_{10}^{+} = (f_0^+f_0^+ - 2f_1^+f_{-1}^+)/2$ and $S_{23}^{+} = (f_2^+f_{-2}^+ - f_3^+f_{-3}^+)/2$ are $SU_2^a \times SU_2^b \times SU_2^c$ invariants. Let

$$\begin{aligned}|2\rangle &= |ww;\, aa;\, \tfrac{1}{2}(w-a)-k,\, \tfrac{1}{2}(w-a)-k,\, \tfrac{1}{2}(w-a)-k\rangle \\ &= N_2\Lambda_{ka}^{w} f_1^{+a} f_3^{+w-a-2k}|0\rangle.\end{aligned} \tag{11a}$$

Using the relation $S_-|2\rangle = 0$, we get

$$\Lambda_{ka}^{w} = \sum_t (-1)^t \binom{k}{t} \frac{(2w-2a-2k+2)!!(2a+1)!!}{(2w-2a-2k-2t+2)!!(2a+2t+1)!!} S_{10}^{+t} S_{23}^{+k-t} \tag{11b}$$

and

$$N_2 = \left(\frac{(w-a-2k+1)(2a+2k+1)!!(2w-2k+3)!!}{k!a!(w-a-k+1)!(2w+3)!!(2a+1)!!}\right)^{1/2}. \tag{11c}$$

Hence the state $|2\rangle$ is a simultaneous eigenstate of the operators C_7, a^2, a_0, b^2, b_0, c_0 with eigenvalues $\frac{1}{2}w(w+5)$, $a(a+1)$, a, $[\frac{1}{2}(w-a)-k][\frac{1}{2}(w-a)-k+1]$, $\frac{1}{2}(w-a)-k$, $\frac{1}{2}(w-a)-k$.

Finally, acting $a_-^{a-a_0}$, $b_-^{(1/2)(w-a)-k-b_0}$, $c_-^{(1/2)(w-a)-k-c_0}$ on the state $|2\rangle$, we obtain a general state vector

$$
\begin{aligned}
|ww; aa_0; \tfrac{1}{2}(w-a)-k\, b_0c_0\rangle
&= N\Lambda_{ka}^{w} \sum_{\sigma\tau} F_{a_0b_0c_0}^{\sigma\tau} f_{-1}^{+\sigma} f_0^{+a-a_0-2\sigma} f_1^{+a_0+\sigma} f_2^{+b_0-c_0+\tau} \\
&\quad\times f_{-2}^{+\tau} f_{-3}^{+(1/2)(w-a)-k-b_0-\tau} f_3^{+(1/2)(w-a)-k+c_0-\tau}|0\rangle
\end{aligned} \tag{12a}
$$

where

$$
\begin{aligned}
F_{a_0b_0c_0}^{\sigma\tau} &= \begin{pmatrix} \tfrac{1}{2}(w-a)-k-b_0 \\ \tau \end{pmatrix} \\
&\quad\times \frac{(a-a_0)!\,a!\,(w-a-2k)!\,(\tfrac{1}{2}(w-a)-k-c_0)!}{(a-a_0-2\sigma)!\,(a_0+\sigma)!\,(2\sigma)!!\,(b_0-c_0+\tau)!}\, \frac{1}{(\tfrac{1}{2}(w-a)-k+c_0-\tau)!}
\end{aligned} \tag{12b}
$$

and

$$
\begin{aligned}
N &= \left(\frac{(a+a_0)!\,[\tfrac{1}{2}(w-a)-k+b_0]!\,[\tfrac{1}{2}(w-a)-k+c_0]!\,2^{a-a_0}}{(a-a_0)!\,(2a)!\,[\tfrac{1}{2}(w-a)-k-b_0]!\,[\tfrac{1}{2}(w-a)-k-c_0]!}\right)^{1/2} \\
&\quad\times [N_2/(w-a-2k)!](-1)^{a-a_0+(1/2)(w-a)-k-b_0}.
\end{aligned} \tag{12c}
$$

4. Physical state

In nuclear physics, for example, in order to describe octopole vibrations in nuclei, the usual IBM (s-d) model is extended to include the f boson [13, 14]. The $SO_7 \supset SO_3$ chain is very important in these applications. In this case, the basis vectors can be written as

$$
|\Psi IM\rangle = [|N\chi\rangle_c \times |n_f L_f\rangle]_M^I \tag{13}
$$

where the subscript c denotes the positive-parity core of active s and d bosons, while $|n_f L_f\rangle$ denotes the $SO_7 \supset SO_3$ basis vectors. In the following we will focus our attention on the construction of the basis vectors for the f-boson system for the $SO_7 \supset SO_3$ chain.

The states constructed in § 3 do not have good angular momentum, and as such are not physical states. In practical calculations this problem may be solved by requiring the states to be eigenstates of the angular momentum L^2 [12]. We will give a method used to construct the basis vectors for $SO_7 \supset SO_3$ as shown below.

Firstly, we consider the highest-weight state

$$
|ww; 00; \tfrac{1}{2}w\, \tfrac{1}{2}w\, \tfrac{1}{2}w\rangle = \sqrt{1/w!}\, f_3^{+w}|0\rangle \tag{14}
$$

which corresponds to the state with $L = 3w$ and $M = 3w$.

Then we act on (14) with L_-, which gives a linear combination of the states with the same w and M. The angular momentum operators are defined as

$$
L_0 = a_0 + 5b_0 + c_0 = \sqrt{28}(f^+\tilde{f})_0^{(1)} \tag{15a}
$$

$$
L_\pm = \sqrt{6}(a_\pm + c_\pm) - \sqrt{10}\, T^{1\frac{1}{2}\frac{1}{2}}_{\pm1\pm\frac{1}{2}\mp\frac{1}{2}} = \sqrt{28}(f^+\tilde{f})_{\pm1}^{(1)}. \tag{15b}
$$

We also have

$$
\begin{aligned}
&T^{1\frac{1}{2}\frac{1}{2}}_{1-\frac{1}{2}\frac{1}{2}}|ww; aa_0; bb_0c_0\rangle \\
&\quad= \frac{1}{4}\left(\frac{(w-a-2b)(w+a+2b+5)(a+a_0+1)(a+a_0+2)(b-b_0+1)(b+c_0+1)}{(2b+1)(2a+1)(2a+3)(b+1)}\right)^{1/2}
\end{aligned}
$$

$$\times|ww;\, a-1\, a_0+1;\, b+\tfrac{1}{2}\, b_0-\tfrac{1}{2}\, c_0+\tfrac{1}{2}\rangle$$

$$-\frac{1}{4}\left(\frac{(w-a-2b+2)(w+a+2b+3)(a-a_0)(a-a_0-1)(b-b_0)(b-c_0)}{(2b+1)(2a+1)(2a-1)b}\right)^{1/2}$$

$$\times|ww;\, a-1\, a_0+1;\, b-\tfrac{1}{2}\, b_0-\tfrac{1}{2}\, c_0+\tfrac{1}{2}\rangle$$

$$+\frac{1}{4}\left(\frac{(w-a+2b+4)(w+a-2b+1)(a-a_0)(a-a_0-1)(b-b_0+1)(b+c_0+1)}{(2b+1)(2a+1)(2a-1)(b+1)}\right)^{1/2}$$

$$\times|ww;\, a-1\, a_0+1;\, b+\tfrac{1}{2}\, b_0-\tfrac{1}{2}\, c_0+\tfrac{1}{2}\rangle$$

$$-\frac{1}{4}\left(\frac{(w-a+2b+2)(w+a-2b+3)(a+a_0+1)(a+a_0+2)(b+b_0)(b-c_0)}{(2b+1)(2a+1)(2a+3)b}\right)^{1/2}$$

$$\times|ww;\, a+1\, a_0+1;\, b-\tfrac{1}{2}\, b_0-\tfrac{1}{2}\, c_0+\tfrac{1}{2}\rangle. \qquad (16)$$

For simplicity, we only discuss the $w=3$ case as an example; in this case the highest state is $f_3^{+3}|0\rangle=|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\rangle$, which corresponding to the physical state with $L=9$ and $M=9$. Acting on $|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\rangle$ with L_- and using (15) and (16), we have $L_-|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}\tfrac{3}{2}\rangle=-3|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}\rangle$, which is the physical state with $L=9$ and $M=8$. Similarly, acting on $|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}\rangle$ with L_-, we obtain

$$L_-|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}\tfrac{1}{2}\rangle=2\sqrt{3}|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}-\tfrac{1}{2}\rangle+\sqrt{5}|3;\,11;\,111\rangle \qquad (17)$$

the state on the RHS corresponding to the state with $L=9$ and $M=7$. The state with $L=7$ and $M=7$ can now be obtained by its normalisation and its orthogonality with $|L=9\ M=7\rangle$:

$$|L=7\ M=7\rangle=2\sqrt{3}|3;\,11;\,111\rangle-\sqrt{5}|3;\,00;\,\tfrac{3}{2}\tfrac{3}{2}-\tfrac{1}{2}\rangle \qquad (18)$$

which is unnormalised and determined within a phase factor.

We can use this method to obtain all the states with good angular momentum quantum numbers.

5. Reduced matrix elements

In this section we will calculate the matrix elements of $T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}$ and some simple isoscalar factors for $SO_7\supset(SU_2)^3$.

First, by acting with $T^{1\frac{1}{2}\frac{1}{2}}_{-1\frac{1}{2}\frac{1}{2}}$ on (12a) for $c_0=b_0$, we obtain

$$T^{1\frac{1}{2}\frac{1}{2}}_{-1\frac{1}{2}\frac{1}{2}}|w;\,aa_0;\,bb_0b_0\rangle$$

$$=\frac{1}{4}\left(\frac{(w-a-2b)(w+a+2b+5)(a-a_0+1)(a-a_0+2)(b+b_0+1)^2}{(2b+1)(2a+1)(2a+3)(b+1)}\right)^{1/2}$$

$$\times|w;\, a+1\, a_0-1;\, b+\tfrac{1}{2}\, b_0+\tfrac{1}{2}\, b_0+\tfrac{1}{2}\rangle$$

$$+\frac{1}{4}\left(\frac{(w-a-2b+2)(w+a+2b+3)(a+a_0)(a+a_0-1)(b-b_0)^2}{(2b+1)(2a+1)(2a-1)b}\right)^{1/2}$$

$$\times|w;\, a-1\, a_0-1;\, b-\tfrac{1}{2}\, b_0+\tfrac{1}{2}\, b_0+\tfrac{1}{2}\rangle$$

$$+\frac{1}{4}\left(\frac{(w-a+2b+4)(w+a-2b+1)(a+a_0)(a+a_0-1)(b+b_0+1)^2}{(2b+1)(2a+1)(2a+3)b}\right)^{1/2}$$

$$\times|w;\, a+1\, a_0-1;\, b-\tfrac{1}{2}\, b_0+\tfrac{1}{2}\, b_0+\tfrac{1}{2}\rangle. \qquad (19)$$

We also have

$$
\begin{aligned}
b_-|w;\, aa_0;\, bb_0c_0\rangle &= -\sqrt{\tfrac{1}{2}(b+b_0)(b-b_0+1)}\,|w;\, aa_0;\, bb_0-1\, c_0\rangle \\
c_-|w;\, aa_0;\, bb_0c_0\rangle &= -\sqrt{\tfrac{1}{2}(b+c_0)(b-c_0+1)}\,|w;\, aa_0;\, bb_0\, c_0-1\rangle \\
a_-|w;\, aa_0;\, bb_0c_0\rangle &= -\sqrt{\tfrac{1}{2}(a+a_0)(a-a_0+1)}\,|w;\, aa_0-1;\, bb_0c_0\rangle.
\end{aligned}
\tag{20}
$$

Using these results and Wigner–Eckart theorem, we obtain the following reduced matrix elements of $T^{1\frac{1}{2}\frac{1}{2}}_{\alpha\beta\gamma}$:

$$
\begin{aligned}
\langle wa+1\; b+\tfrac{1}{2}\| T^{1\frac{1}{2}\frac{1}{2}} \| wab\rangle &= \tfrac{1}{2}[(w+a+2b+5)(w-a-2b)(2b+1)(2b+2)(a+1)]^{1/2} \\
\langle wa+1\; b-\tfrac{1}{2}\| T^{1\frac{1}{2}\frac{1}{2}} \| wab\rangle &= \tfrac{1}{2}[(w-a+2b+2)(w+a-2b+3)(2b+1)(2b)(a+1)]^{1/2}.
\end{aligned}
\tag{21}
$$

The above results are the same as given by [5]. Furthermore, acting with f_3^+ on (12*a*) we get

$$
\langle w+1\; a\, b+\tfrac{1}{2}\| f^+ \| wab\rangle
= [(w-a+2b+4)(2b+1)(2b+2)(2a+1)(w+a+2b+5)/2(2w+5)]^{1/2}. \tag{22a}
$$

Similarly, acting with f_1^+ on (12*a*), we obtain

$$
\langle w+1\; a+1\; b\| f^+ \| wab\rangle
= [(w+a-2b+3)(w+a+2b+5)(a+1)(2b+1)^2/(2w+5)]^{1/2}. \tag{22b}
$$

We know that f_3^+ is simultaneously a rank-1 tensor of SO_7, $\frac{1}{2}$-spinor of SU_2^b and SU_2^c and scalar of SU_2^a, the isoscalar factor for $SU_7 \supset SO_7$ $\langle ww11\| w+1 w+1\rangle = 1$ and $\langle w+1 \||f^+\|| w\rangle = \sqrt{w+1}$, where $\langle w+1\||f^+\||w\rangle$ is a SU_7 reduced matrix element. Using these results, we obtain the $SO_7 \supset (SU_2)^3$ isoscalar factors as shown below:

$$
\begin{aligned}
\langle wab-\tfrac{1}{2}10\tfrac{1}{2}\| w+1\, ab\rangle &= \left(\frac{(w-a+2b+3)(w+a+2b+4)b}{(2w+5)(w+1)(2b+1)}\right)^{1/2} \\
\langle wa-1\, b110\| w+1\, ab\rangle &= \left(\frac{(w+a-2b+2)(w+a+2b+4)a}{(2w+5)(w+1)(2a+1)}\right)^{1/2} \\
\langle wa+1\, b110\| w+1\, ab\rangle &= \left(\frac{(a+1)(w-a-2b+1)(w-a+2b+3)}{(2w+5)(w+1)(2a+1)}\right)^{1/2} \\
\langle wab+\tfrac{1}{2}10\tfrac{1}{2}\| w+1\, ab\rangle &= \left(\frac{(b+1)(w-a-2b+1)(w+a-2b+2)}{(2w+5)(w+1)(2b+1)}\right)^{1/2}.
\end{aligned}
\tag{23}
$$

By using the reciprocity relation

$$
\begin{aligned}
&\langle w+1\, ab1\| wa'b'\rangle \\
&\quad = (-1)^{a'-\alpha-a+2b'-2b-2\beta+\gamma}\left(\frac{(2a+1)(2b+1)^2(2w+5)(w+1)}{(2a'+1)(2b'+1)^2(2w+7)(w+5)}\right)^{1/2} \\
&\quad\quad \times \langle wa'b'1\| w+1\, ab\rangle
\end{aligned}
\tag{24}
$$

the isoscalar factors $\langle wa'b'1\|w-1\,ab\rangle$ can also be obtained; the results are summarised as follows:

$$\begin{aligned}
\langle wab-\tfrac{1}{2}10\tfrac{1}{2}\|w-1\,ab\rangle &= \left(\frac{(w-a-2b+1)(w+a-2b+1)b}{(2w+5)(w+4)(2b+1)}\right)^{1/2}\\
\langle wa-1\,b110\|w-1\,ab\rangle &= \left(\frac{(w-a-2b+1)(w-a+2b+3)a}{(2w+5)(w+4)(2a+1)}\right)^{1/2}\\
\langle wa+1\,b110\|w-1\,ab\rangle &= \left(\frac{(w+a-2b+2)(w+a+2b+4)(a+1)}{(2w+5)(w+4)(2a+1)}\right)^{1/2}\\
\langle wab+\tfrac{1}{2}10\tfrac{1}{2}\|w-1\,ab\rangle &= \left(\frac{(w-a+2b+3)(w+a+2b+4)(b+1)}{(2w+5)(w+4)(2b+1)}\right)^{1/2}
\end{aligned} \tag{25}$$

where we simply choose $y=0$.

6. Summary

In this paper we have used the method outlined in [12] to construct the basis vectors for the symmetric irreps of $SO_7 \supset (SU_2)^3$. It can be seen that the missing-label problem can be solved by using the mathematical basis, and that the explicit basis vectors can be constructed by using the boson operator technique (BOT). In addition, this technique is easier than other methods [5] for deriving the reduced matrix elements and the isoscalar factors. By using the angular momentum projection procedure, the eigenstates of the angular momentum L^2 can be constructed with these basis vectors.

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